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CONTENTS

ABSTRACT

The growth of a vapor bubble in a superheated liquid is controlled by three factors: the inertia of the liquid, the surface tension, and the vapor pressure. As the bubble grows, evaporation takes place at the bubble boundary, and the temperature and vapor pressure in the bubble are thereby decreased. The heat inflow requirement of evaporation, however, depends on the rate of bubble growth, so that the dynamic problem is linked with a heat diffusion problem. Since the heat diffusion problem has been solved, a quantitative formulation of the dynamic problem can be given. A solution for the radius of the vapor bubble as a function of time is obtained which is valid for sufficiently large radius. This asymptotic solution covers the range of physical interest since the radius at which it becomes valid is near the lower limit of experimental observation. It shows the strong effect of heat diffusion on the rate of bubble growth. Comparison of the predicted radius-time behavior is made with experimental observations in superheated water, and very good agreement is found.

I. INTRODUCTION

When the vapor pressure in a liquid exceeds the ambient pressure, it becomes possible for a vapor bubble to grow from a small "nucleus" in the liquid. This nucleus is a region of nonliquid phase and presumably consists of a gas or vapor phase stabilized on a solid particle. The rate of growth of a vapor bubble, once formed, is determined by the surface tension, the liquid inertia, and the difference between the pressure within the bubble and the ambient, or external, pressure. In the initial stages of the bubble expansion, when the forces are nearly in equilibrium, the growth is slow but it is accelerated with increase in bubble size because of the reduction in surface tension. When the rate of bubble growth becomes appreciable, however, the temperature and hence the pressure within the bubble drop and the rate of growth is decreased. One might, therefore, expect a maximum in the velocity of the bubble wall. The reduction of the temperature within the bubble is a consequence of the latent heat requirement of the evaporation which takes

place at the vapor-liquid interface as the bubble grows.

For the quantitative solution of the problem, some simplifying physical assumptions may be made. The example considered in detail here is that of a vapor bubble growing in moderately superheated water, and the arguments for the validity of the general assumptions are justified for this case.

It will be assumed that the bubble is spherical throughout its growth. This assumption is reasonable provided the radial acceleration and velocity are small, for with a spherically symmetric external pressure field the spherical shape will then be stable under the action of surface tension. [1]

Excluded from consideration is the asymmetric, buoyant force of gravity which becomes important if the bubble growth is followed for so long a time that a significant translational velocity is acquired. A translational velocity of the bubble as a whole not only causes a deformation in shape but also increases the rate of heat inflow to the bubble over that used in the analysis presented below. In water superheated by about $10^\circ C$, no great error is introduced by the buoyant force provided the bubble growth is not followed beyond a radius of approximately 1 mm.

For a superheat of $10^\circ C$ in water, a vapor bubble grows from its initial microscopic size to a radius of 1 mm in a time of the order of 10 millisec. The corresponding average radial velocity of 10 cm/sec is very small compared with the velocity of sound in the liquid, so that compressibility effects may be entirely neglected in the motion of the liquid produced by the bubble growth. It may also be shown that viscous effects are unimportant. [2] The hydrodynamic equation of motion, which contains the effect of the liquid inertia, is thus greatly simplified. The bubble wall velocity is also very small compared with the velocity of sound in the vapor so that compressibility effects are unimportant here as well. One has the further simplification in the vapor region that the vapor density is so small that its inertia effects may be neglected. It then follows from Bernoulli's equation applied to a fluid of negligible density that the pressure within the vapor region may be taken as uniform. In addition, since the acoustic velocity in the vapor is so large, the pressure within the vapor follows practically instantaneously its value at the bubble wall. When the

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[1] M. S. Plesset, J. Appl. Phys., in press.

[2] F. R. Gilmore, Report 26-4, Hydrodynamics Laboratory, California Institute of Technology, April 1952.

velocity of the bubble wall is sufficiently slow, the pressure of the vapor is given by the equilibrium vapor pressure of the liquid. That this is the case for the present problem may be seen as follows. The mean velocity appropriate for the rate of evaporation from a liquid surface is $(B T/2\pi M)^{1/2}$ where B is the gas constant, T the absolute temperature, and M is the molecular weight. This velocity must be reduced by the coefficient for evaporation which has the value 0.04 for water.[3] The critical velocity for a water surface at approximately $100^\circ C$ is therefore about 3 m/sec, which is appreciably greater than the radial velocities encountered here, so that the pressure deficiency from the equilibrium vapor pressure may be neglected.

The temperature of the vapor would, in general, vary with position in the bubble as well as with time. The approximation will be made, however, that the thermal diffusivity D' of the vapor is so large that temperature gradients are negligible within the bubble. In water vapor the characteristic diffusion length $(2D't)^{1/2}$ is 0.24 cm for $t = 10^{-2}$ sec. For the greatest superheat considered here ($\approx 10^\circ C$) the bubble radius at this time is approximately 0.1 cm, which is significantly smaller than the characteristic diffusion length. The approximation of uniform temperature within the vapor improves with decrease in superheat.

In summary, the physical model upon which the calculations are based consists of a spherical vapor bubble which has uniform temperature and pressure; the temperature of the vapor is that of the liquid at the bubble wall, and the pressure is the equilibrium vapor pressure for that temperature. In addition, the effects of viscosity and compressibility are neglected both in the vapor and in the liquid.

The equation of motion for the radius R of the bubble in a nonviscous, incompressible liquid as a function of time is[4]

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = [p(R) - P_0]/\rho ,\qquad (1)$$

where ρ is the liquid density, P_0 is the external pressure in the liquid or the pressure at infinity, and $p(R)$ is the pressure in the liquid at the bubble

[3]G. Wyllie, Proc. Roy. Soc. (London) A 197, 333 (1949).

[4]H. Lamb, Hydrodynamics (Dover Pub. New York, 1945); M. S. Plesset, J. Appl. Mech. 16, 277 (1949).

boundary. A dot denotes differentiation with respect to time. The pressure $p(R)$ is given in terms of the vapor pressure p_v within the bubble by

$$p(R) = p_v(T) - 2\sigma/R , \qquad (2)$$

where σ is the surface tension constant and $p_v(T)$ is the equilibrium vapor pressure for the temperature T at the bubble boundary. In the following, the small variations of σ and ρ with temperature are neglected. It is convenient to introduce a radius R_o defined by the relation

$$2\sigma/R_o = p_v(T_o) - P_o , \qquad (3)$$

in which T_o is the temperature of superheat of the liquid at a distance from the bubble. Physically, R_o is the effective initial radius of the bubble; it represents an extrapolation of the free spherical bubble down to the equilibrium radius for the given initial conditions. It should be noted that a bubble at rest with radius R_o is in unstable equilibrium. The actual nucleus from which the bubble grows is not necessarily spherical and its surface energy may be appreciably less than $4\pi\sigma R_o^2$; however, the nucleus from which the bubble grows and the free spherical bubble of radius R_o are both in unstable equilibrium at the temperature T_o and external pressure P_o. Table I gives a set of values of R_o at various superheat temperatures in water for a pressure of P_o of one atm.

In terms of the parameter R_o, Eq. (1) may be written

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{p_v(T) - p_v(T_o) + (2\sigma/R_o)(1 - R_o/R)}{\rho} , \qquad (4)$$

or

$$\frac{1}{2R^2\dot{R}} \frac{d}{dt} (R^3\dot{R}^2) = \frac{p_v(T) - p_v(T_o) + (2\sigma/R_o)(1 - R_o/R)}{\rho} , \qquad (5)$$

If the cooling effect of evaporation is disregarded, so that $p_v(T) = p_v(T_o)$, Eq. (5) may be integrated to give

$$\dot{R}^2 = (R_o/R)\dot{R}_o^2 + (4\sigma/3\rho R_o)\left[1 - R_o^3/R^3\right] - (2\sigma/\rho R)\left[1 - R_o^2/R^2\right] . \qquad (6)$$

This solution will be referred to as the Rayleigh solution.[5] For $R \gg R_0$, Eq. (6) becomes

$$\dot{R}^2 \sim \frac{4\sigma}{3\rho R_0} = \frac{2}{3\rho}\left[P_v(T_0) - P_0 \right],$$ (7)

which is a constant.

The actual motion deviates markedly from that predicted by the Rayleigh solution because of the cooling effect. The heat $\dot{Q}$ which must be supplied to the bubble per unit time is

$$\dot{Q} = (4\pi/3)L \frac{d}{dt}(R^3 \rho'),$$ (8)

where L is the latent heat of evaporation per unit mass and ρ' is the vapor density. This heat is supplied by conduction from the liquid into the bubble so that

$$\dot{Q} = 4\pi R^2 k(\partial T/\partial r)_R,$$ (9)

where k is the thermal conductivity of the liquid and $(\partial T/\partial r)_R$ is the temperature gradient in the liquid at the bubble boundary. Hence,

$$(\partial T/\partial r)_R = \frac{L}{3k} \frac{1}{R^2} \frac{d}{dt}(R^3 \rho').$$ (10)

The small variations of L and k with temperature are neglected. It may be shown that the contribution of the term proportional to $d\rho'/dt$ is much less than that from the term proportional to dR^3/dt. Equation (10) may be approximated, therefore, by

$$(\partial T/\partial r)_R \approx (L\rho'/k)\dot{R}.$$ (11)

This approximation is plausible physically since the volume increases by orders of magnitude, while the vapor density changes very little because of the small temperature decrease during the bubble growth.

Equation (11) together with the specification of the temperature at

[5] The solution for the motion of a bubble under such constant pressure conditions was given by Rayleigh, Phil. Mag. 34, 94 (1917) and applied by him to the case of a collapsing bubble.

infinity T_0 determines the temperature field in the liquid. The solution for this temperature problem with the moving boundary $R(t)$ has been found[6] under the assumption that the drop in temperature from T_0 to the value T at the bubble boundary takes place in a layer of liquid adjacent to the bubble which has small thickness compared with $R(t)$. This approximation of the "thin thermal boundary layer" is justified physically because the thermal diffusivity of the liquid is small. The approximate expression for the temperature at the bubble wall is[7]

$$T = T_0 - \left(\frac{D}{\pi}\right)^{1/2} \int_0^t \frac{R^2(x)(\partial T/\partial r)_{r=R(x)}}{\left[\int_x^t R^4(y)\,dy\right]^{1/2}} \, dx \; . \tag{12}$$

The temperature T is thus given in terms of R and $\dot{R}$ by Eq. (12) so that $p_v(T)$ is also specified in terms of these variables. Equations (5) and (12) therefore determine the dynamic problem of the bubble growth.

II. SOLUTION OF THE EQUATION OF MOTION

Equations (5) and (12) are connected when the equilibrium vapor pressure is specified as a function of temperature. For superheats not too far above the boiling temperature T_b of the liquid at the external pressure P_0, the vapor pressure may be approximated by a linear function of the temperature:

$$\frac{p_v(T) - P_0}{\rho} = A(T - T_b) \; . \tag{13}$$

For $T = T_0$, Eq. (13) gives

$$2\sigma/\rho R_0 = A(T_0 - T_b) \tag{14}$$

because of Eq. (3).

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[6]M. S. Plesset and S. A. Zwick, J. Appl. Phys. **23**, 95 (1952).

[7]Ref. 6, Eq. (20). The error in Eq. (20) may be estimated from Eq. (31) of this reference. For the problem discussed here it was found to be less than 10% of the difference $T - T_0$ at any time.

Equation (5) may now be written as

$$\frac{1}{2R^2\dot{R}} \frac{d}{dt} (R^3\dot{R}^2) = A(T - T_0) + \frac{2\sigma}{\rho R_0} (1 - \frac{R}{R_0}) . \qquad (15)$$

It is convenient to use in place of R the dimensionless volume ratio

$$p = R^3/R_0^3 , \qquad (16a)$$

and in place of t the dimensionless variable

$$u = (a/R_0^4) \int_0^t R^4(y) dy , \qquad (16b)$$

where the constant a is defined as

$$a = \left[\frac{2\sigma}{\rho R_0^3}\right]^{1/2} \qquad (16c)$$

and has the dimensions of reciprocal time. Equation (15) then is transformed to

$$\frac{1}{6p'} \frac{d}{du} (p^{7/3} p'^2) = 1 - \frac{1}{p^{1/3}} - \mu \int_0^u \frac{p'(v) dv}{(u-v)^{1/2}} , \qquad (17)$$

in which $p' = dp/du$ and the dimensionless parameter μ is given by

$$\mu = \frac{AL_0'}{3\pi R_0 a} \left[\frac{D}{\pi a}\right]^{1/2} . \qquad (18)$$

The physical quantities of interest are now expressed as follows:

$$t = \frac{1}{a} \int_0^u \frac{dv}{p^{4/3}(v)} \quad ; \tag{19a}$$

$$R = R_o p^{1/3} \quad ; \tag{19b}$$

$$\dot{R} = (a R_o/3) \, p^{2/3} p' \quad ; \tag{19c}$$

$$T - T_o = - \frac{a^2 R_o^2 \mu}{A} \int_0^u \frac{p'(v) \, dv}{(u - v)^{1/2}} \quad . \tag{19d}$$

The values of the radius R for which experimental observations are readily obtainable are much greater than R_o so that the asymptotic solution of Eq. (17) ($p \gg 1$) is of the greatest physical interest. It is of importance however, to examine the initial stages of the bubble growth since a qualitative description of this portion of the problem is necessary for an understanding of the asymptotic solution. It will be shown that the solution in the asymptotic range is not affected by the details of the mathematical model used to describe the behavior of the bubble for R near R_o so that the uncertainty in the surface energy for small values of R causes no difficulties in the physical applications to be made here.

A. Initial Growth of the Bubble

It is evident from Eq. (17) that a bubble of radius R_o ($p = 1$) initially at rest ($p'(0) = 0$) will remain at rest in unstable equilibrium. The present mathematical model requires some change in the external conditions before bubble growth will be initiated.

The particular mechanism which will be considered here is a rise in the temperature of the liquid. Suppose, for example, that there is a constant heat source of strength $\underline{a}$ per unit volume so that the amount of heat added per unit volume η from $t = 0$ to $t = t$ is

$$\eta = at \quad . \tag{20}$$

Then, if the temperature of the bulk liquid is T_o at $t = 0$, at time t it will be

$$T_{\infty} = T_0 + \eta/\rho c = T_0 + \eta D/k \ . \tag{21}$$

The heat flow problem may be solved[6] for the boundary conditions of Eqs. (21) and (10), and the temperature at the bubble wall is given by replacing T_0 in Eq. (12) by T_{∞}. In place of Eq. (19d) one has

$$T = T_0 + \eta D/k - \frac{a^2 R_0^2 \mu}{A} \int_0^u \frac{p'(v)\, dv}{(u-v)^{1/2}} \ . \tag{22}$$

The extension of Eq. (17) to include the heat source term is thus

$$\frac{1}{6p'} \frac{d}{du} (p^{7/3} p^2) = 1 - \frac{1}{p^{1/3}} - \mu \int_0^u \frac{p'(v)\, dv}{(u-v)^{1/2}} + \frac{AD}{a^2 R_0^2 k} \eta \ . \tag{23}$$

For values of p near the initial value of unity, one may write

$$\eta = at = \frac{a}{\alpha} \int_0^u \frac{dv}{p^{4/3}(v)} \approx \frac{a}{\alpha} u \ ,$$

where the substitution for t follows from Eq. (19a). Equation (23) now becomes

$$\frac{1}{6p'} \frac{d}{du} (p^{7/3} p^2) = 1 - \frac{1}{p^{1/3}} - \mu \int_0^u \frac{p'(v)\, dv}{(u-v)^{1/2}} + \gamma u \ , \tag{23'}$$

where

$$\gamma = \frac{AD a}{a^3 R_0^2 k} \ . \tag{24}$$

The constant γ is extremely small; for example, in water with a temperature rise of $1°C/sec$ and $R_0 = 10^{-3}$ cm, γ is approximately 10^{-6}. The smallness of the constant γ implies that the forced growth away from the unstable equilibrium point $p = 1$, $p' = 0$ is very slow until the bubble radius has increased sufficiently for the surface tension to be partially "relaxed". This initial slow growth is a delay period in the bubble growth, for the

bubble radius changes very little until the remaining terms on the right side of Eq. (23') become appreciable. The bubble growth is then so rapid that the change in the bulk temperature of the liquid is insignificant, and the term in γ may be neglected. The delay period effectively gives an initial value of R slightly greater than R_o from which the important growth begins. An approximate solution may be found for this initial period of forced growth from the equilibrium point by linearizing Eq. (23'); that is, by neglecting terms of the second order in $p-1$ and its derivatives. The details of this calculation will not be presented here; but, as would be expected, the forced growth away from the equilibrium has an exponential behavior:

$$R \approx R_o \left[1 + \frac{2\gamma}{\beta(3\beta^2 + 1)} e^{\alpha\beta t} \right] . \tag{25}$$

The temperature at the bubble wall is given approximately by

$$T \approx T_\infty - \left[\frac{R_o^2 \alpha^2}{A} \right] \frac{2\gamma(1-\beta^2)}{\beta(3\beta^2 + 1)} e^{\alpha\beta t} . \tag{25}$$

The constant β is that root of the equation

$$\beta^2 + 3\mu(\pi\beta)^{1/2} - 1 = 0$$

for which $\beta^{1/2}$ is positive.

B. Growth Behavior for $R >)R_o$

After the bubble growth has been initiated, there is a rapid rise in the velocity $\dot{R}$ until the cooling effect becomes important. The bubble wall velocity decreases continuously thereafter. No observations on bubble growth in this region are as yet available and the details of this analysis will be omitted. Of present interest is the asymptotic phase of the bubble growth determined by Eq. (17) which is characterized by the limiting effects on $\dot{R}$ of the heat diffusion from the liquid to the vapor. As R increases, the temperature at the bubble wall decreases steadily, but it cannot fall below T_b; for, if the vapor temperature f below T_b, the pressure difference $p_v - P_o$ would become negative and t bubble growth would be arrested and eventually reversed. Such

behavior is excluded on physical grounds. It therefore follows that the integral on the right hand side of Eq. (17), which is proportional to the temperature drop at the bubble wall, must approach a limit as t or $u \to \infty$. A further physical argument determines more precisely the asymptotic behavior of this integral: The left hand side of Eq. (17) represents essentially the acceleration effects of the bubble growth in the liquid. As the bubble grows, this acceleration tends toward zero because of the cooling effect. It therefore follows that

$$\mu \int_0^u \frac{p'(v)\,dv}{(u-v)^{1/2}} \sim 1 \qquad \text{as } u \to \infty \,. \tag{27}$$

This asymptotic relation may be inverted to yield[8]

$$p(u) \sim \frac{2}{\pi\mu}\, u^{1/2} \qquad\qquad \text{as } u \to \infty \,. \tag{28}$$

If this result is substituted in Eq. (17), it may be verified that Eqs. (17), (27) and (28) are, in fact, consistent.

Equation (28) is not yet useful since it provides no means of matching the indicated asymptotic solution of Eq. (17) with a solution valid for small values of p. The possibility of matching solutions depends on the possibility of shifting the asymptotic solution in t (or in u) so as to account for the delay period in bubble growth. It is necessary that one be free to shift the asymptotic solution since the duration of the delay period depends completely on the choice of the heat source term while the subsequent behavior of the bubble is independent of this term. A means for making an arbitrary time shift is furnished by noting that, in addition to the asymptotic solution of Eq. (28), Eq. (17) also possesses the solution $p(u) = 1$. It will therefore be assumed that the asymptotic solution is described by

$$p(u) = 1, \qquad 0 < u \leq u_1 \,, \tag{29}$$

$$\mu \int_{u_1}^u \frac{p'(v)\,dv}{(u-v)^{1/2}} = 1 - \frac{1}{p^{1/3}} - \frac{1}{6p'}\frac{d}{du}(p^{7/3}p^2), \quad u > u_1 \,. \tag{30}$$

---

[8]Equation (27) is multiplied by $(z-u)^{-1/2}$ and integrated from $u = 0$ to $u = z$ to give Eq. (28).

From Eq. (19a), one has corresponding to Eqs. (29) and (30)

$$
t = \frac{1}{a} \int_0^u \frac{dv}{p^{4/3}(v)} =
\begin{cases}
\dfrac{u}{a}, & 0 < u \leqslant u_1, \\[3ex]
\dfrac{u_1}{a} + \dfrac{1}{a} \displaystyle\int_{u_1}^u \dfrac{dv}{p^{4/3}}, & u > u_1,
\end{cases}
\tag{31}
$$

so that u_1/a represents the duration of the delay period in the growth. The time delay may be introduced explicitly in the asymptotic solution for $u > u_1$ by use of the fact that, if $p(u)$ is a solution of Eq. (30), $p(u+u_o)$ is also a solution where u_o is a constant. A consistent scheme for continuing the asymptotic solution may then be found by taking the solution to be of the form

$$
p(u) = 1, \qquad 0 < u \leqslant u_1
\tag{32a}
$$

$$
p(u) \sim \frac{2}{\pi \mu} (u - u_o)^{1/2} \left[1 + \frac{b_1}{(u - u_o)^{1/6}} + \cdots + \frac{b_5}{(u - u_o)^{5/6}} + \frac{b_6 \, \ell n(u - u_o)}{(u - u_o)} \right],
$$

$$
u > u_1
\tag{32b}
$$

to seven terms.[9] When the coefficients b_k have been determined, the difference $(u_1 - u_o)$ is fixed by the requirement that $p(u_1) = 1$. The delay period is then determined by the choice of u_o.

When (32b) is substituted in the integral on the left side of Eq. (30) the result is[10]

$$
\mu \int_{u_1}^u \frac{p'(v) \, dv}{(u - v)^{1/2}} \sim 1 + \frac{2 \Gamma(2/6)}{3\pi^{1/2} \Gamma(5/6)} \frac{b_1}{(u - u_o)^{1/6}} + \frac{\Gamma(1/6)}{3\pi^{1/2} \Gamma(4/6)} \frac{b_2}{(u - u_o)^{2/6}}
$$

$$
+ \left(\frac{2}{\pi} b_3 - \mu\right) \frac{1}{(u - u_o)^{3/6}} - \frac{\Gamma(-1/6)}{3\pi^{1/2} \Gamma(2/6)} \frac{b_4}{(u - u_o)^{4/6}} - \frac{2\Gamma(-2/6)}{3\pi^{1/2} \Gamma(1/6)} \frac{b_5}{(u - u_o)^{5/6}} + 2 \frac{b_6}{(u - u_o)}.
$$

$$
\tag{33}
$$

---

[9]Higher terms are of the form $\left[\ell n(u - u_o) \right]^m /(u - u_o)^{n/6}$, where m and n are integers.

[10]See the appendix for the evaluation of the integral.

By Eq. (30), this expression is also asymptotic to

$$1 - \frac{1}{p^{1/3}} - \frac{1}{6p'} \frac{d}{du} (p^{7/3} p'^2) \, . \tag{34}$$

If (34) is expanded by (32b), the coefficients of corresponding powers of $(u - u_o)$ may be equated to give a set of successive equations for the parameters $b_1, b_2, \ldots b_6$. At each step, one has a linear equation for the unknown parameter. A tabulation of these parameters for various superheat conditions in water for an external pressure of 1 atm. is given in Table II.

The leading terms in the asymptotic solution are

$$p = (R/R_o)^3 \sim \frac{2}{\pi \mu} u^{1/2} \left[1 + 0(u^{-1/6}) \right] ;$$

$$t \sim \frac{3}{a} \left(\frac{\pi \mu}{2} \right)^{4/3} u^{1/3} \left[1 + 0(u^{-1/6}) \right] ;$$

$$T - T_o \sim - \frac{a^2 R_o^2}{A} \left[1 + 0(u^{-1/6}) \right] .$$

Thus,

$$R \sim R_o \left(\frac{2}{\pi \mu} \frac{at}{3} \right)^{1/2} \left[1 + 0(t^{-1/2}) \right] ; \tag{35}$$

$$T - T_o \sim - \frac{a^2 R_o^2}{A} \left[1 + 0(t^{-1/2}) \right] . \tag{36}$$

The vapor density ρ' may be approximated by a linear function of the temperature over the small range of present concern. Consequently

$$\rho' \sim \text{const.} \cdot \left[1 + 0(t^{-1/2}) \right] \qquad \text{as } t \longrightarrow \infty ,$$

by Eq. (36). The volume of the bubble, on the other hand, behaves like

$$R^3 \sim \text{const.} \cdot t^{3/2} \left[1 + 0(t^{-3/2}) \right] \qquad \text{as } t \longrightarrow \infty .$$

It is evident that the vapor density varies slowly in the asymptotic range compared with the bubble volume so that the neglect of $d\rho'/dt$ compared with dR^3/dt in the formulation of the boundary condition of Eq. (11) is justified.

III. COMPARISON WITH EXPERIMENT AND CONCLUSION

With the values of the constants given in Table II, radius-time curves have been computed for water at 1 atm for superheats in the range of $102^\circ C$ to $106^\circ C$. These curves are presented in Fig. 1. Inasmuch as the delay period may be chosen arbitrarily so far as these asymptotic solutions are concerned, the time scale is determined only within an arbitrary constant which varies from one curve to another. The actual spacing of the curves as presented was chosen so that the time intercepts at $R = 0.04$ cm were equally spaced.

Observations have been made by Dergarabedian[11] on the growth of vapor bubbles in superheated water. The comparison between the theoretical curves and the observed values is shown in Figs. 2, 3, and 4. The theoretical curves were obtained by interpolation from the values graphed in Fig. 1. The time origins for both the theoretical curves and the experimental points are both arbitrary so that a time translation of the theoretical curve has been made in each case to give the best fit. The agreement is, however, seen to be very good. The importance of the cooling effect is evident from Fig. 2 where the Rayleigh solution (Eq. (7)) is also shown.

Equation (35) gives for the leading term in the asymptotic velocity

$$\dot{R} \sim \frac{R_o}{\pi \mu} \left(\frac{a}{\bar{J}t}\right)^{1/2} = \left(\frac{3}{\pi}\right)^{1/2} \frac{k(T_o - T_b)}{L \rho' D^{1/2}} \frac{1}{t^{1/2}} . \tag{37}$$

While the number of terms given in Eq. (32b) is sufficient for high accuracy in the range of values of R used in Figs. 1 - 4, the leading term in Eq. (32b) is in error by approximately 10 to 30% for the smallest values of R. This leading term is sufficiently simple, however, that a rough physical argument should give the essential variation of Eq. (37). The argument is as follows: At a time t at which the bubble radius R is much greater than R_o, the difference between the temperature in the liquid at the bubble wall and that in the liquid at a distance is only slightly less than $T_o - T_b$. This temperature drop takes place principally in a liquid layer around the bubble of approximate thickness given by the diffusion length $(D t)^{1/2}$. The heat flow into the bubble

[11] P. Dergarabedian, J. Appl. Mech., in press.

per unit time is therefore given roughly by

$$\dot{Q} \approx k \frac{(T_o - T_b)}{(Dt)^{1/2}} \, 4\pi R^2 \quad . \tag{38}$$

The heat requirement per unit time of evaporation, on the other hand is

$$\dot{Q} = L \frac{d}{dt} \left(\frac{4}{3} \pi R^3 \rho' \right) \approx 4\pi R^2 \dot{R} L \rho' \quad . \tag{39}$$

When (39) is equated to (38), there results

$$\dot{R} \sim \frac{k(T_o - T_b)}{L \rho' (Dt)^{1/2}} \quad . \tag{40}$$

which agrees in order of magnitude with the leading term as given by Eq. (37).

Some experiments have recently been performed by Dergarabedian on vapor bubble growth in pure CCl_4. At moderate superheats, its vapor pressure curve is approximately parallel to that of water, displaced to lower temperatures. If the rates of bubble growth in these two liquids are compared at the same value of the temperature difference $(T_o - T_b)$, they should thus be roughly in the same ratio as $k(L \rho' D^{1/2})$ for the two liquids. This constant is about 3.3 greater in water than in carbon tetrachloride. Dergarabedian's observations on bubble growths are in good agreement with this value.

APPENDIX

Evaluation of the Temperature Integral (Eq. (33))

By differentiating $p(u)$ (Eq. (32b)) and substituting into the temperature integral there results, after a change of variable,

$$\mu \int_{u_1}^{u} \frac{p'(v)\, dv}{(u-v)^{1/2}}$$

$$\sim \frac{1}{\pi} \int_{\left(\frac{u_1-u_o}{u-u_o}\right)}^{1} \frac{dv}{v^{1/2}(1-v)^{1/2}} \left\{ 1 + \left[\frac{2}{3}\frac{b_1}{(u-u_o)^{1/6}} \right] \frac{1}{v^{1/6}} + \left[\frac{1}{3}\frac{b_2}{(u-u_o)^{2/6}} \right] \frac{1}{v^{2/6}} \right.$$

$$ - \left[\frac{1}{3}\frac{b_4}{(u-u_o)^{3/6}} \right] \frac{1}{v^{4/6}} - \left[\frac{2}{3}\frac{b_5}{(u-u_o)^{5/6}} \right] \frac{1}{v^{5/6}}$$

$$ \left. - \left[\frac{b_6}{u-u_o} \right] \left[\ln v + \ln(u-u_o) - 2 \right] \frac{1}{v} \right\} . \qquad (a)$$

Consider a typical integral appearing in (a),

$$I_s(x) = \int_x^1 \frac{v^{s-1}\, dv}{(1-v)^{1/2}} . \qquad (b)$$

For $0 < x < 1$, $\mathrm{Re}(s) > 0$, this is the hypergeometric function[*]

$$I_s(x) = \frac{1}{s} \left[F(1/2,\ s;\ s+1;\ 1) - x^s F(1/2,\ s;\ s+1;\ x) \right] , \qquad (c)$$

$$\sim \frac{\Gamma(1/2)\Gamma(s)}{\Gamma(s+1/2)} - \frac{1}{s} x^s \qquad \text{as } x \to 0^+ . \qquad (d)$$

By an argument based on the theory of analytic continuation, however, it may readily be shown that (c) is valid provided only that $0 < x < 1$, $s \neq 0, -1, -2, \ldots,$ and hence that (d) holds for $0 < x < 1$, $\mathrm{Re}(s) > -1$. For $s = 0$, (d) is

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[*] E. T. Copson, Introduction to Theory of Functions of a Complex Variable, (Oxford Univ. Press, London), p. 249, ...

meaningless, but gives a correct result under a limiting procedure. By differentiating (d) with respect to s at $s = -1/2$, one readily finds

$$\int_x^1 \frac{\ell n v \, dv}{v^{3/2}(1-v)^{1/2}} \sim -2\pi + \frac{2}{x^{1/2}}(\ell n x + 2) \quad \text{as} \quad x \to 0^+ \tag{e}$$

With the aid of (d), (e), Eq. (a) therefore becomes

$$\mu \int_{u_1}^u \frac{p'(v) \, dv}{(u-v)^{1/2}}$$

$$\sim \left\{ 1 + .89266 \, \frac{b_1}{(u-u_o)^{1/6}} + .77306 \, \frac{b_2}{(u-u_o)^{2/6}} \right.$$

$$\left. + .47545 \, \frac{b_4}{(u-u_o)^{4/6}} + .27450 \, \frac{b_5}{(u-u_o)^{5/6}} + 2 \, \frac{b_6}{(u-u_o)} \right\}$$

$$- \frac{2}{\pi} \frac{1}{(u-u_o)^{1/2}} (u_1-u_o)^{3/6} \left\{ 1 + \frac{b_1}{(u_1-u_o)^{1/6}} + \frac{b_2}{(u_1-u_o)^{2/6}} + \frac{b_4}{(u_1-u_o)^{4/6}} \right.$$

$$\left. + \frac{b_5}{(u_1-u_o)^{5/6}} + b_6 \, \frac{\ell n(u_1-u_o)}{u_1-u_o} \right\} \tag{f}$$

after a rearrangement, several terms cancelling. The Γ-functions appearing in (d) have been evaluated numerically in (f). By comparison with the asymptotic form for $p(u)$ of Eq. (30) the last term in (f) may be written

$$\frac{1}{(u-u_o)^{1/2}} \qquad \frac{2}{\pi} b_3 - \mu p(u_1) \qquad .$$

Inasmuch as $p(u_1) = 1$ by assumption, Eq. (f) reduces to Eq. (33) of the text.

TABLE I. Effective nucleus size, R_o, in water at one atmosphere for the values of the temperature T_o.

R_o (cm×10³)	1.56	1.02	0.751	0.590	0.403
T_o (°C)	102	103	104	105	106

TABLE II. Values of the parameters for water at various superheat temperatures. The external pressure P_o = 1 atm; the constant A has been given the value 4.08 × 10⁴ dynes cm/(°C gm).

| T	102°	103°	104°	105°	106°
R_o	1.558×10^{-3}	1.019×10^{-3}	7.507×10^{-4}	5.901×10^{-4}	4.832×10^{-4}
a	1.797×10^{5}	3.391×10^{5}	5.356×10^{5}	7.677×10^{5}	1.035×10^{6}
μ	1.6794	1.0234	7.2215×10^{-1}	5.5244×10^{-1}	4.4414×10^{-1}
b_1	-1.07324	-9.09893×10^{-1}	-8.10074×10^{-1}	-7.40873×10^{-1}	-6.88097×10^{-1}
b_2	-4.70929×10^{-1}	-4.62444×10^{-1}	-7.12168×10^{-1}	-1.33372	-2.50569
b_3	2.33920×10^{-1}	-1.48106×10^{-3}	-3.50564×10^{-1}	-9.72510×10^{-1}	-1.97166
b_4	-5.55386×10^{-1}	1.25839×10^{-4}	-1.97013×10^{-1}	-4.59239	-2.50901×10^{-1}
b_5	-1.77478	3.06358×10^{-1}	4.29791	1.85037×10^{-1}	6.58379×10^{-1}
b_6	-7.42262×10^{-1}	-2.67045×10^{-1}	6.16637×10^{-1}	-4.73640×10^{-1}	-4.05527×10^{-1}
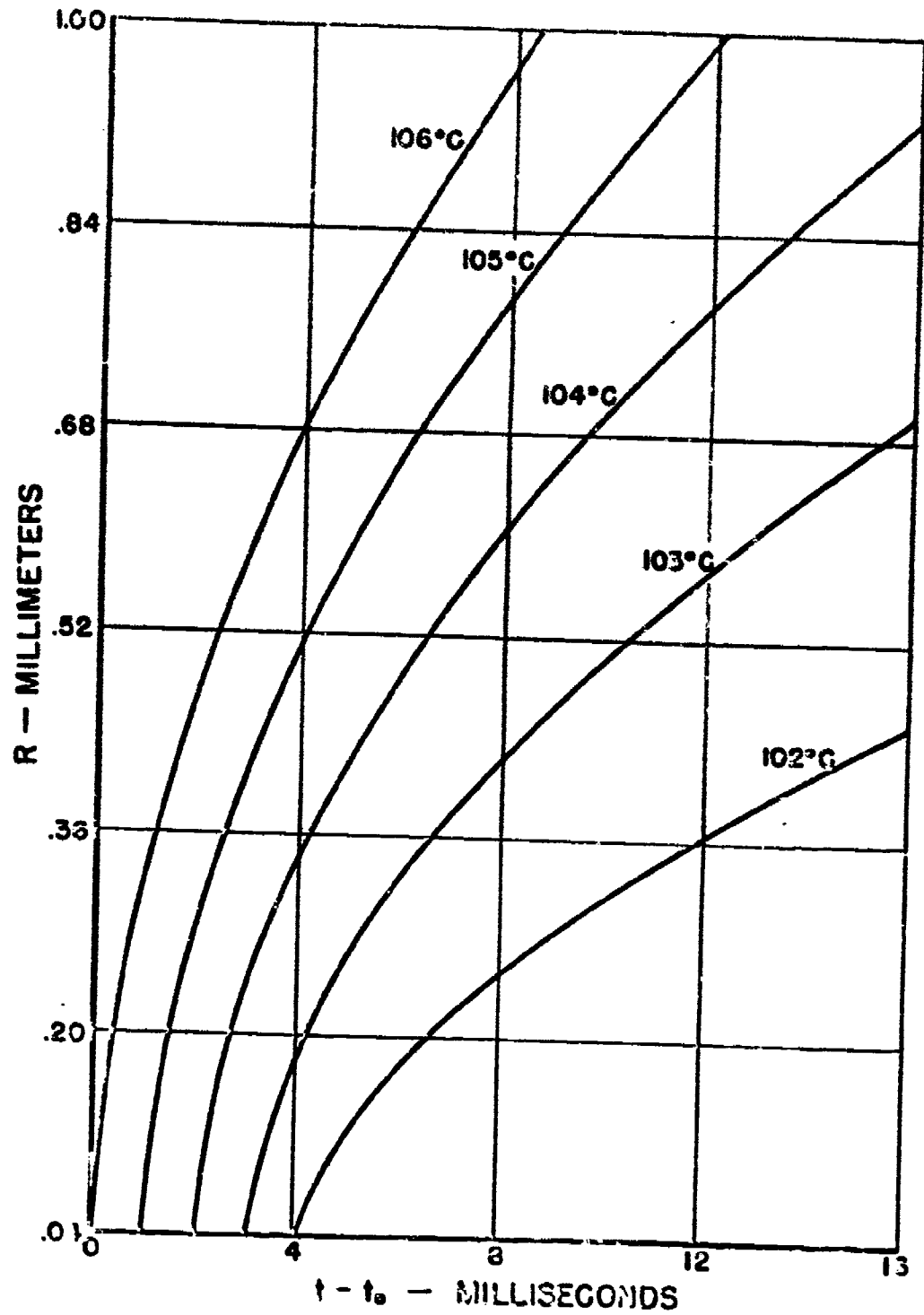
Fig. 1 - Calculated bubble radius versus time curves at indicated superheat temperatures in water at 1
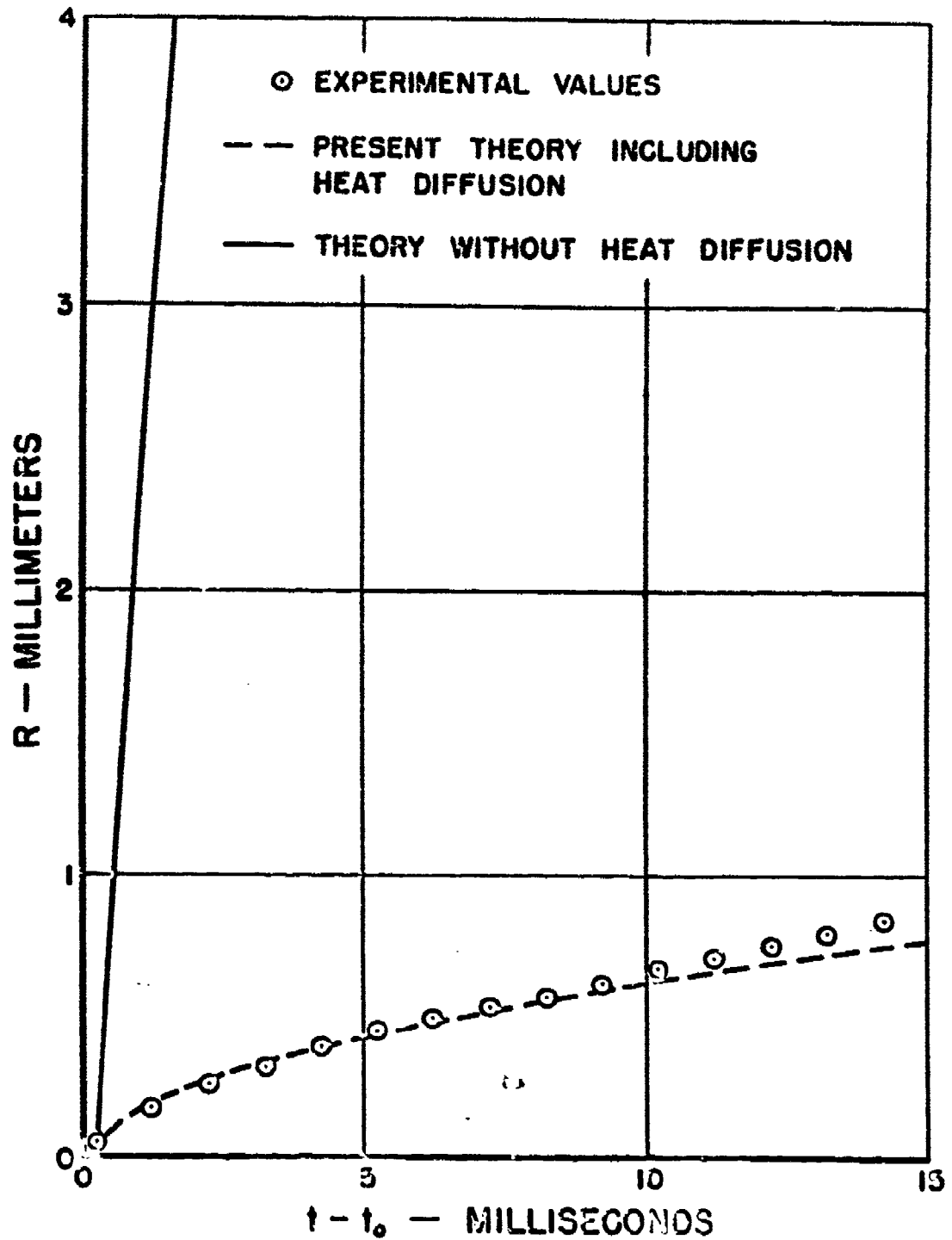
Fig. 2 – Comparison of theoretical bubble radius-time values with experimental values in water at 1 atm with a superheat temperature of 103.1°C. The solid curve is the Rayleigh growth curve neglecting the cooling effect; the dashed curve includes this effect by the method indicated in the text.
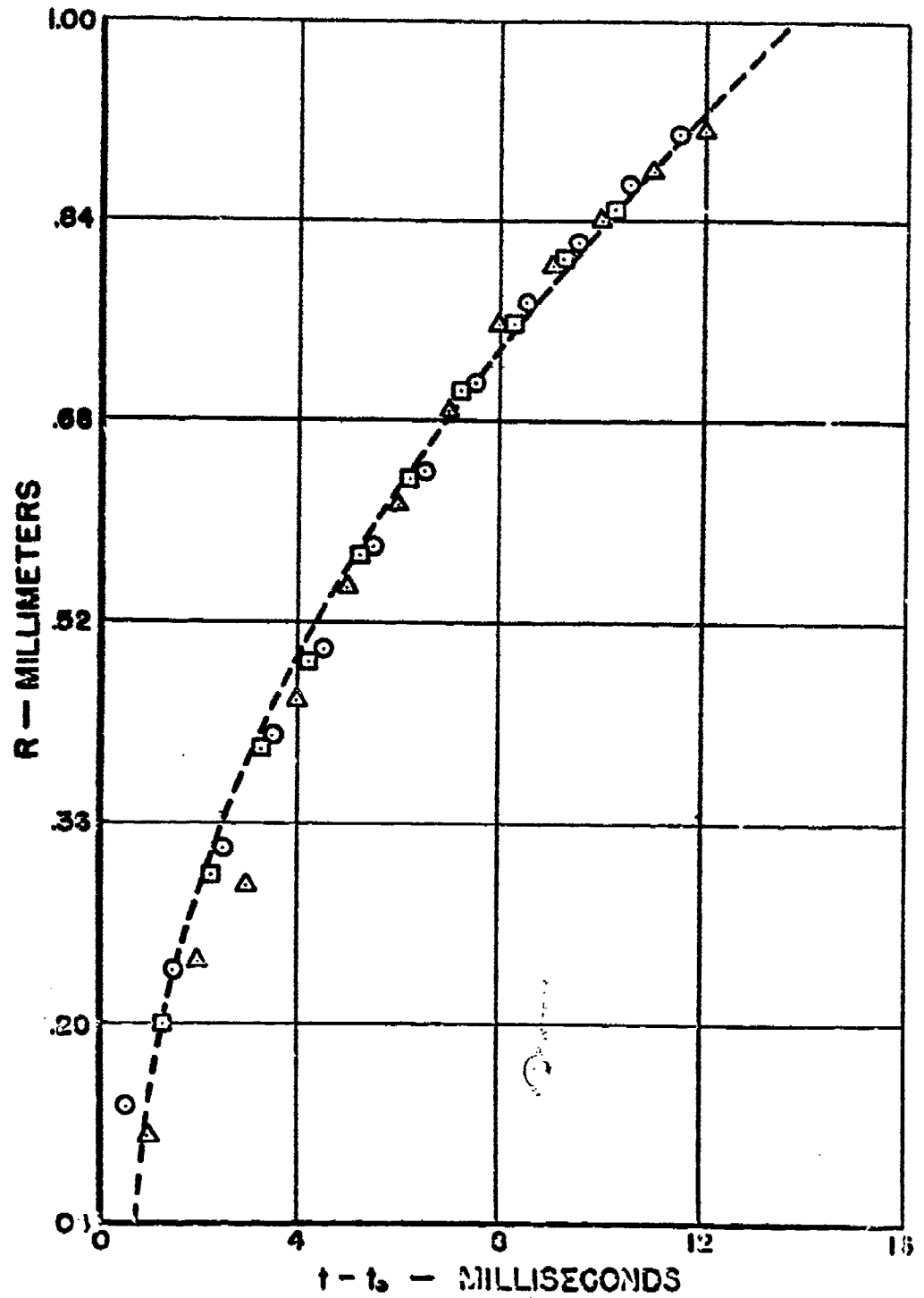
Fig. 3 - Comparison of theoretical radius-time values with three sets of experimental values obtained in water superheated to 104.5°C at an external pressure of 1 atm.

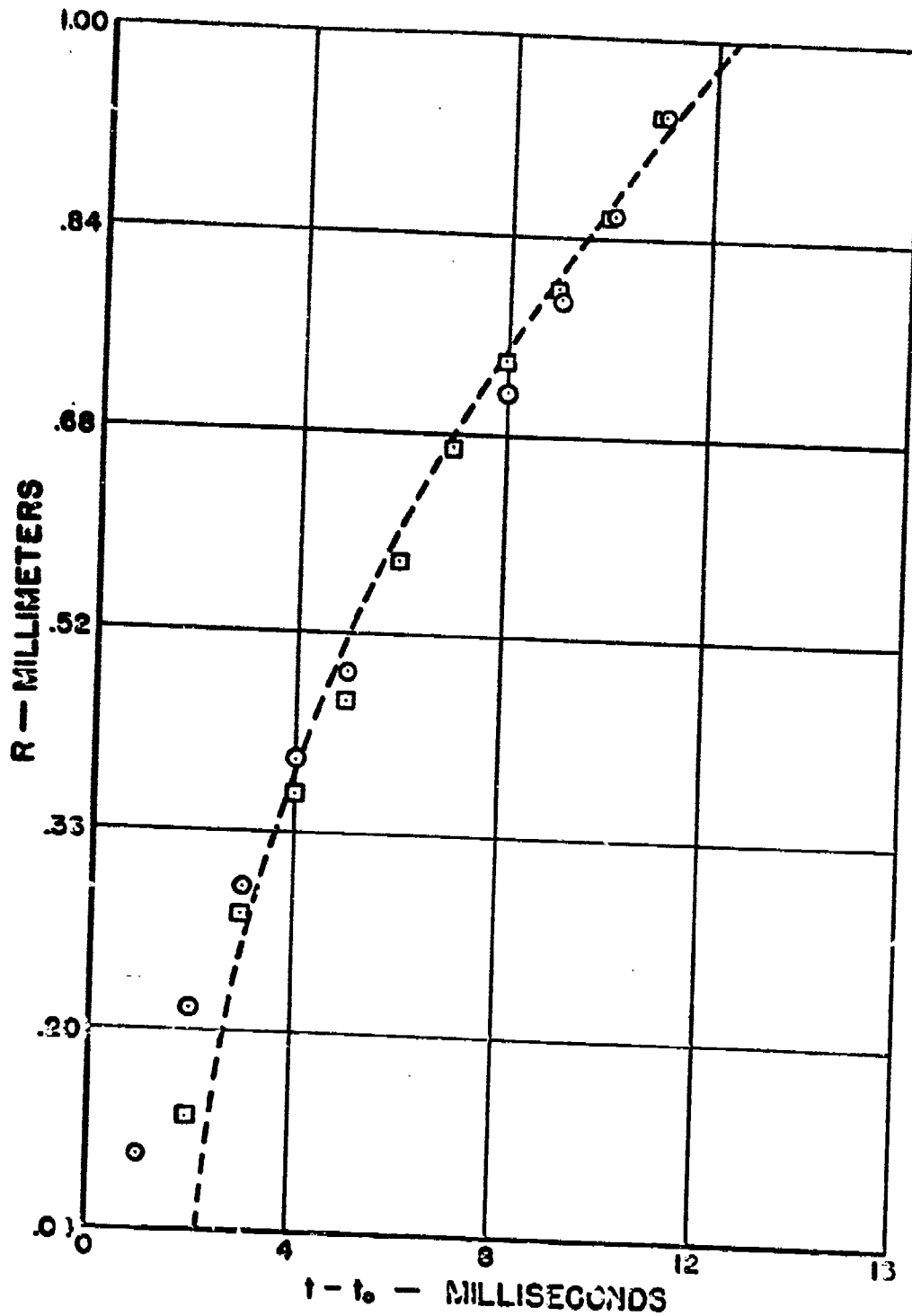**Fig. 4 -** Comparison of theoretical radius-time values with two sets of experimental values obtained in water superheated to 105.3°C at an external pressure of 1 atm.